

# Density of the $\mathrm{SO}(3)$ TQFT representation of mapping class groups

by

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## §0. Introduction

A (2+1)-dimensional topological quantum field theory (TQFT) determines, for each  $g \geq 0$ , a projective representation  $(\rho_g, V_g)$  of the mapping class group  $M_g$  of a closed oriented surface of genus  $g$ . This paper is concerned with the  $\mathrm{SO}(3)$  TQFT at an  $r$ th root of unity,  $r \geq 5$  prime. Those TQFTs were first constructed mathematically in [T]. The problem we consider is this: what is the closure of  $\rho_g(M_g)$ ?

For  $g = 1$  and the  $\mathrm{SU}(2)$ -theory, Kontsevich observed that the image is finite. A proof of finiteness both for the  $\mathrm{SU}(2)$ -theory and the  $\mathrm{SO}(3)$ -theory may be found in [G]; see also [J], for an early calculation from which the finiteness can be deduced. For  $g \geq 2$ , the image was shown to be infinite, and its closure therefore of positive dimension [F]. In this paper, we identify the representation for  $g = 1$  and show that the image is either  $\mathrm{SL}_2(\mathbb{F}_r)$  or  $\mathrm{PSL}_2(\mathbb{F}_r)$  depending on whether  $r$  is congruent to 1 or  $-1 \pmod{4}$ . For  $g \geq 2$ , on the other hand, we show that the image is as large as possible—that is, it is dense in the group of projective unitary transformations on the representation space of  $M_g$ .

We are working with the  $\mathrm{SO}(3)$ -theory because the statements are cleaner than for the  $\mathrm{SU}(2)$ -theory. Using the density result here and the tensor decomposition formulas in Theorem 1.5 [BHMV], the closed images of the  $\mathrm{SU}(2)$ -theory can be identified. The restriction to a prime  $r$  is dictated by a result of Roberts [R] of which we make essential use: the  $\mathrm{SU}(2)$ -theory representations of  $M_g$  are irreducible for  $r$  prime.

The case  $r = 3$  is trivial since  $\dim V_g = 1$  for all  $g$ . The case  $r = 5$  was treated in joint work with Freedman [FLW2]. Our proof depended on the observation that a Dehn twist acts with  $\frac{r-1}{2}$  distinct eigenvalues. For  $r = 5$ , this means that the  $M_g$  representations satisfy the “two-eigenvalue property.” The compact Lie groups  $G$  admitting a representation

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with respect to which a generating conjugacy class has only two eigenvalues can be classified, and the few possibilities can be reduced to one by an examination of the branching rules for the restriction  $M_{g-1} \subset M_g$  and dimension computations (especially the Verlinde formula). For  $r \geq 7$ , we no longer have the two-eigenvalue property, and the number of possibilities for  $G$  grows rapidly with  $r$ . Mainly, therefore, we depend on the branching rules. A crucial point is to prove that the representations are tensor-indecomposable, i.e., not equivalent to a tensor product of representations of lower degrees; this is precisely why the  $\mathrm{SO}(3)$  case is simpler than that of  $\mathrm{SU}(2)$ . The reason tensor-indecomposability is so important is that, coupled with irreducibility, it implies that the identity component of the closure of  $\rho_g(M_g)$  is a simple group, and this greatly shortens the list of possibilities.

The original motivation for this work was topological quantum computation in the sense of [FKLW], [FLW1], and [FLW2]. As in [FLW2], there are also applications to the distribution of values of 3-manifold invariants. As a simple example, we show that the set of the norms of the Witten-Reshetikhin-Turaev  $\mathrm{SO}(3)$  invariants of all connected 3-manifolds at  $A = ie^{\frac{2\pi i}{4r}}$  is dense in  $[0, \infty)$  for primes  $r \geq 5$ .

## §1. The $\mathrm{SO}(3)$ -TQFT

There are several constructions of the  $\mathrm{SU}(2)$  and the  $\mathrm{SO}(3)$  TQFTs in the literature (e.g., [BHMV][FK][RT][T]). The  $\mathrm{SU}(2)$  TQFT was first constructed mathematically in [RT], and the  $\mathrm{SO}(3)$  TQFT in [T]. We will follow Turaev's book [T], where the construction of a TQFT is reduced to the construction of a modular tensor category.

Fixing a prime  $r \geq 5$  and setting  $A = ie^{\frac{2\pi i}{4r}} = e^{\frac{2\pi i(r+1)}{4r}}$ , note that  $A$  is a primitive  $2r$ -th root of unity when  $r \equiv 1 \pmod{4}$ , and a primitive  $r$ th root of unity when  $r \equiv -1 \pmod{4}$ . In [BHMV] to construct TQFT using the skein theory, the Kauffman variable  $A$  is either a primitive  $4r$ th or a primitive  $2r$ th root of unity. When  $r \equiv 1 \pmod{4}$  by the  $\mathrm{SO}(3)$  TQFT we mean the TQFT denoted by  $V_r$  in [BHMV] with the above choice of  $A$ . When  $r \equiv -1 \pmod{4}$ , the same construction still gives rise to a TQFT although  $A$  is only a primitive  $r$ th root of unity, which is also denoted by  $V_r$  here, but the decomposition formula in Theorem 1.5 [BHMV] does not necessarily hold. Consequently we have to distinguish between the two cases  $r \equiv 1 \pmod{4}$  and  $r \equiv -1 \pmod{4}$ .

The modular tensor categories associated to the TQFTs  $V_r$  are described in [T, Chapter XII]. In particular, [T, Theorem 9.2] discussed the unitarity of the TQFTs. For the above choice of  $A$ 's, the ribbon categories of [T, Theorem 9.2] are not modular because the  $S$ -matrices as given in [T, Lemma 5.2] are singular. (The Kauffman variable  $A$  is a primitive  $4r$ th root of unity for only even  $r$ 's.) But it can be shown that the even subcategories

(see [T, Section 7.5]) are indeed modular and unitary [T][FNWW]. The even subcategories correspond to the restriction of the representation categories to the odd dimensional (or integral spin) “halves” in the quantum group setting [FK].

A modular tensor category consists of a large amount of data. For our purpose here we will only specify the isomorphism classes of simple objects, called *labels* of the associated TQFT, the  $S$ -matrix, and the  $T$  matrix. More information is contained in Lemma 2.

We write the quantum integers  $[k]_A = \frac{A^{2k} - A^{-2k}}{A^2 - A^{-2}}$ . The label set of the  $V_r$  theory is  $L = \{0, 2, 4, \dots, r - 3\}$ . The quantum dimension of the label  $i$  is given by  $d_i = [i + 1]$ , the subscript  $A$  in  $[k]_A$  will be dropped from now on, and the global dimension of the  $V_r$  modular tensor category is  $D = \sqrt{\sum_{i \in L} [i + 1]^2} = \frac{\sqrt{r}}{2 \sin \frac{\pi}{r}}$ . The  $\tilde{S}$ -matrix  $\tilde{S} = (\tilde{s}_{ij})$  can be read off from Lemma 5.2 [T] as  $\tilde{s}_{ij} = [(i + 1)(j + 1)]$ . The  $T = (t_{ij})$  matrix is diagonal with diagonal entries the twists  $\theta_i$ , which are computed in [KL, Proposition 6, p. 43] as  $\theta_i = A^{i(i+2)}$ .

Let  $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ,  $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  be the generators of  $\mathrm{SL}_2(\mathbb{Z})$ . It is a deep fact that the  $S, T$  matrices give rise to a projective matrix representation of  $\mathrm{SL}_2(\mathbb{Z})$  if we make the following assignments:  $s \rightarrow S = \frac{1}{D} \cdot \tilde{S}, t \rightarrow T^{-1}$ , i.e., the  $SO(3)$  TQFT representation  $\rho_{SO(3)}$  for  $\mathrm{SL}_2(\mathbb{Z})$  is:

$$\rho_{SO(3)}(s) = \frac{1}{D}([i + 1][j + 1])$$

and

$$\rho_{SO(3)}(t) = (A^{-j(j+2)} \delta_{ij}).$$

The  $T$ -matrix corresponds geometrically to a Dehn twist, so the negative twists  $\theta_i^{-1}$  are the eigenvalues for the image of any Dehn twist on a non-separating simple closed curve.

*Remark:* It seems to be generally believed that the two theories  $V_r$  and  $V_{2r}$  constructed in [BHMV] correspond to the  $SO(3)$  and the  $SU(2)$  Witten-Reshetikhin-Turaev TQFTs. Actually the  $S$ -matrix of the  $V_{2r}$  theory is not the same as that in the Witten-Reshetikhin-Turaev  $SU(2)$  theory [Wi][RT]: the  $(i, j)$ -th entry differs by a sign  $(-1)^{i+j}$ . But this discrepancy disappears on restriction to the even subcategories; this is the reason that the Witten-Reshetikhin-Turaev TQFTs are always unitary, but the  $V_{2r}$  theories are not unitary in general. This subtle point is due to the Frobenius-Schur indicators for self-dual representations, and will be clarified in [FNWW].

LEMMA 1: *If  $A$  is a primitive  $2r$ th root of unity and  $r \geq 3$  is odd, then there exists a TQFT  $V'_2$  and a natural isomorphism of theories such that*

$$V_{2r}(\Sigma) \cong V'_2(\Sigma) \otimes V_r(\Sigma).$$

*Moreover, the  $\mathrm{SO}(3)$ -theory representations of the mapping class groups  $M_g$  are irreducible for all primes  $r \geq 5$ .*

*Proof:* The decomposition formula is Theorem 1.5 of [BHMV].

To prove the second part, first consider the case  $r \equiv 1 \pmod{4}$ . Suppose the  $\mathrm{SO}(3)$ -theory representation of the mapping class groups for a closed surface is reducible for a prime  $r$ . Then by the tensor decomposition formula above the  $\mathrm{SU}(2)$ -theory representation would be reducible, too. But this contradicts the result of [R]. Therefore, the  $\mathrm{SO}(3)$ -theory representations are also irreducible. For  $r \equiv -1 \pmod{4}$ , the same argument will work if a similar decomposition formula holds. Without such a formula, the irreducibility of the  $\mathrm{SO}(3)$  representations of  $M_g$  can be deduced by following Roberts's argument [R].  $\square$

LEMMA 2: *Let  $d_{r,g,h_1,\dots,h_k}$  denote the dimension of the vector space associated by the  $\mathrm{SO}(3)$  theory at an  $r$ th root of unity to a compact oriented surface  $\Sigma$  of genus  $g$  with  $k$  boundary components labeled  $h_1, \dots, h_k \in 2\mathbb{Z}$ . Then we have the following:*

- 1)  $d_{r,0,h} = \delta_{h0}$ , where  $\delta$  is the Kronecker delta.
- 2)  $d_{r,0,h_1,h_2} = \delta_{h_1 h_2}$ .
- 3)  $d_{r,0,h_1,h_2,h_3}$  is 1 if and only if the  $h_i$  satisfy the triangle inequality (possibly with equality) and  $h_1 + h_2 + h_3 \leq 2(r-2)$ ; otherwise it is 0.
- 4) (Gluing formula) Suppose  $\Sigma$  is cut along a simple closed curve, and  $\Sigma_x$  denotes the resulting surface with the two new boundary components both labeled by  $x$ . Then

$$V_r(\Sigma) \cong \bigoplus_{x \in L} V_r(\Sigma_x),$$

where  $L = \{0, 2, \dots, r-3\}$  is the label set of the  $V_r$  theory.

- 5)  $d_{r,1,h} = \frac{r-1-h}{2}$ .
- 6)  $d_{r,g,h_1,\dots,h_k,0} = d_{r,g,h_1,\dots,h_k}$ .
- 7) (Verlinde formula)  $d_{r,g} = \sum_{j=1}^{\frac{r-1}{2}} \alpha_j^{g-1}$ , where  $\alpha_j = \frac{r \csc^2 \frac{2\pi j}{r}}{4}$ .

*Proof:* Parts 1)–4) are the basic data of the  $V_r$  theory (see also [KL]). Parts 5) and 6) are easy consequences of 1)–4). The Verlinde formula is derived in [BHMV].  $\square$

## §2. Tensor products and decompositions

Next we prove some technical results which enable us to establish that certain representations are tensor indecomposable. We say a complex representation  $V$  of a compact Lie group  $G$  is **isotypic** if it is of the form  $W^n = W \otimes \mathbb{C}^n$  for some irreducible representation  $W$  of  $G$ . If  $W$  is one-dimensional, we say  $V$  is **scalar**. Two representations of  $G$  are **conjugate** if one is equivalent to the composition of the other with an automorphism of  $G$ .

LEMMA 3: *Let  $0 \rightarrow G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow 0$  be a short exact sequence of compact Lie groups and  $\rho : G_2 \rightarrow \mathrm{GL}(V)$  an irreducible representation. If the restriction of  $V$  to  $G_1$  is isotypic, then there exist central extensions  $\tilde{G}_1$  and  $\tilde{G}_2$  of  $G_1$  and  $G_2$  respectively and representations  $\sigma$  and  $\tau$  of  $\tilde{G}_2$  such that*

- (1) *the extension  $\tilde{G}_1$  is a normal subgroup of  $\tilde{G}_2$ ,*
- (2) *the quotient  $\tilde{G}_2/\tilde{G}_1$  is isomorphic to  $G_3$ ,*
- (3) *the restriction of  $\sigma$  to  $\tilde{G}_1$  is irreducible,*
- (4) *the restriction of  $\tau$  to  $\tilde{G}_1$  is scalar, and*
- (5) *the tensor product  $\sigma \otimes \tau$  is equivalent to the composition of  $\rho$  with the central quotient map  $\tilde{G}_2 \rightarrow G_2$ .*

*Proof:* By hypothesis,  $V|_{G_1}$  is isotypic and so can be written as  $W \otimes \mathbb{C}^k$ . The span of  $\rho(G_1)$  is

$$\rho(G_1)\mathbb{C} = \mathrm{End}(W) \subset \mathrm{End}(W \otimes \mathbb{C}^k) = \mathrm{End}(V),$$

where  $\mathrm{End}(W)$  maps to  $\mathrm{End}(W \otimes \mathbb{C}^k)$  by  $x \mapsto x \otimes \mathrm{Id}_k$ . The image  $\rho(G_2)$  lies in the normalizer  $\mathrm{End}(W)\mathrm{End}(\mathbb{C}^k)$  of  $\rho(G_1)\mathbb{C}$ . Thus,  $\rho$  can be regarded as a map

$$G_2 \rightarrow (\mathrm{End}(W)\mathrm{End}(\mathbb{C}^k))^* = (\mathrm{GL}(W) \times \mathrm{GL}_k(\mathbb{C}))/\mathbb{C}^*.$$

Let

$$\tilde{G}_2 = G_2 \times_{(\mathrm{GL}(W) \times \mathrm{GL}_k(\mathbb{C}))/\mathbb{C}^*} (\mathrm{GL}(W) \times \mathrm{GL}_k(\mathbb{C})),$$

be a central extension of  $G_2$ ,  $\tilde{G}_1$  the pre-image of  $G_1$  in  $\tilde{G}_2$  with respect to the central quotient map  $\pi : \tilde{G}_2 \rightarrow G_2$ , and  $\tilde{\rho}$  the pullback  $\tilde{G}_2 \rightarrow (\mathrm{GL}(W) \times \mathrm{GL}_k(\mathbb{C}))$  of  $\rho$ . Let  $\sigma$  and  $\tau$  denote the compositions of  $\tilde{\rho}$  with the projection maps  $\mathrm{GL}(W) \times \mathrm{GL}_k(\mathbb{C}) \rightarrow \mathrm{GL}(W)$  and  $\mathrm{GL}(W) \times \mathrm{GL}_k(\mathbb{C}) \rightarrow \mathrm{GL}_k(\mathbb{C})$  respectively. The diagram

$$\begin{array}{ccccccc} G_1 & & \leftarrow & \tilde{G}_1 & \rightarrow & \tilde{G}_2 & \\ \downarrow & & & \downarrow & & \downarrow \tilde{\rho} & \\ GL(W) = (\mathrm{GL}(W) \times \mathbb{C}^*)/\mathbb{C}^* & \leftarrow & \mathrm{GL}(W) \times \mathbb{C}^* & \rightarrow & \mathrm{GL}(W) \times \mathrm{GL}_k(\mathbb{C}) & & \end{array}$$

shows that the restrictions of  $\sigma$  and  $\tau$  to  $\tilde{G}_1$  are irreducible and scalar respectively, and (5) is immediate.  $\square$

In the other direction, we have:

LEMMA 4: *Let  $G_2$  be a compact Lie group and  $G_1$  a closed normal subgroup. Let  $V$  and  $W$  be irreducible representations of  $G_2$  such that  $V|_{G_1}$  is irreducible and  $W|_{G_1}$  is scalar. Then  $V \otimes W$  is irreducible.*

*Proof:* As  $V$  and  $W$  are irreducible,

$$\dim \text{End}_{G_2}(V) = \dim(V \otimes V^*)^{G_2} = 1; \quad \dim \text{End}_{G_2}(W) = \dim(W \otimes W^*)^{G_2} = 1.$$

Let

$$V \otimes V^* = \bigoplus_i V_i, \quad W \otimes W^* = \bigoplus_j W_j$$

denote decompositions into irreducible  $G_2$ -representations, numbered so that  $V_1$  and  $W_1$  are trivial (and therefore the other  $V_i$  and  $W_j$  are non-trivial). Thus,

$$\text{End}_{G_2}(V \otimes W) = ((V \otimes V^*) \otimes (W \otimes W^*))^{G_2} = \bigoplus_{i,j} (V_i \otimes W_j)^{G_2} = \bigoplus_{\{i,j \mid V_i \cong W_j^*\}} \mathbb{C}.$$

Now,  $W_j|_{G_1}$  is trivial for all  $j$ , so  $V_i \cong W_j^*$  implies that  $V_i|_{G_1}$  is trivial. However,

$$1 = \dim \text{End}_{G_1}(V) = \dim(V \otimes V^*)^{G_1} = \sum_i \dim V_i^{G_1},$$

so this is possible only for  $i = 1$ . Then  $W_j \cong V_1^*$  implies  $j = 1$ , so

$$\dim \text{End}_{G_2}(V \otimes W) = 1.$$

$\square$

LEMMA 5: *For  $r \geq 5$ , the tensor product of any two non-trivial irreducible representations of  $\text{SL}_2(\mathbb{F}_r)$  has an irreducible factor of degree  $> \frac{r-1}{2}$ .*

*Proof:* This can be deduced from the character table [Sp] p. 160, whose notation we follow. Let  $\chi_1$  and  $\chi_2$  be non-trivial irreducible characters of  $\text{SL}_2(\mathbb{F}_r)$ . If

$$\chi_1 \chi_2 = e \chi_\beta^+ + f \chi_\beta^- + g,$$

then

$$g = \begin{cases} 1 & \text{if } \chi_2 = \bar{\chi}_1, \\ 0 & \text{otherwise.} \end{cases}$$

If  $d_b \in T_1$  is a square, then  $\chi_\beta^{\pm 1}(d_b) = -1$ . If  $\chi_2 = \bar{\chi}_1$ , comparing values at  $d_b$ ,  $e + f \leq 1$ , which is absurd. Thus  $g = 0$ , and thus  $|\chi_1(d_b) \chi_2(d_b)| = e + f$ . As  $\chi(1) \geq \frac{r-1}{2} |\chi(d_b)|$  for all non-trivial  $\chi$ , we get a contradiction.  $\square$

LEMMA 6: *Consider a short exact sequence of compact Lie groups*

$$0 \rightarrow G_1 \rightarrow G_2 \rightarrow \mathrm{PSL}_2(\mathbb{F}_r) \rightarrow 0,$$

where  $r \geq 7$ . If  $H_2$  is the inverse image of a proper subgroup  $H \subset \mathrm{PSL}_2(\mathbb{F}_r)$ , and  $V$  is a representation of  $H_2$ , then some irreducible factor of  $\mathrm{Ind}_{H_2}^{G_2} V$  has degree  $> \frac{r-1}{2}$ .

*Proof:* Without loss of generality, we may assume that  $V$  is irreducible. Let  $K = \ker G_1 \rightarrow \mathrm{GL}(\mathrm{Ind}_{H_2}^{G_2} V)$ . Replacing  $G_2$  and  $H_2$  by  $G_2/K$  and  $H_2/K$  respectively if necessary, we may assume the restriction of  $\mathrm{Ind}_{H_2}^{G_2} V$  to  $G_1$  is faithful. Suppose that the restriction of some irreducible factor  $W$  of  $\mathrm{Ind}_{H_2}^{G_2} V$  to  $G_1$  fails to be isotypic. By Clifford's theorem,  $W$  is a direct sum of mutually conjugate isotypic representations  $W_i^k$  of  $G_1$ ; the stabilizer of  $W_1^k$  can be regarded as a subgroup  $\Delta$  of  $\mathrm{PSL}_2(\mathbb{F}_r)$ , and

$$\dim W = k \dim W_1[\mathrm{PSL}_2(\mathbb{F}_r) : \Delta].$$

By the well-known classification of subgroups of  $\mathrm{PSL}_2(\mathbb{F}_r)$ , each proper subgroup has index  $> \frac{r-1}{2}$ , so  $\dim W > \frac{r-1}{2}$ . We may therefore assume that the center of  $G_1$  is in the center of  $G_2$ . By a theorem of Eilenberg and Mac Lane [EM], the obstruction to finding a section of  $G_2 \rightarrow \mathrm{PSL}_2(\mathbb{F}_r)$  lies in  $H^2(\mathrm{PSL}_2(\mathbb{F}_r), Z(G_1))$ . As  $\mathrm{PSL}_2(\mathbb{F}_r)$  is perfect with universal central extension  $\mathrm{SL}_2(\mathbb{F}_r)$ ,  $G_2$  contains a subgroup  $\Gamma$  isomorphic to  $\mathrm{PSL}_2(\mathbb{F}_r)$  or  $\mathrm{SL}_2(\mathbb{F}_r)$  which maps onto  $\mathrm{PSL}_2(\mathbb{F}_r)$ . As

$$\mathrm{Res}_{G_2}^{\Gamma} \mathrm{Ind}_{H_2}^{G_2} V = \mathrm{Ind}_{H_2 \cap \Gamma}^{\Gamma} \mathrm{Res}_{G_2}^{\Gamma} V,$$

we can reduce to the case that  $G_2$  is  $\mathrm{PSL}_2(\mathbb{F}_r)$  or  $\mathrm{SL}_2(\mathbb{F}_r)$ . If  $G_2 = \mathrm{PSL}_2(\mathbb{F}_r)$  and  $\tilde{H}_2$  is the preimage of  $H_2 \subset G_2$  in  $\mathrm{SL}_2(\mathbb{F}_r)$ , then  $\mathrm{Ind}_{H_2}^{G_2} V$ , regarded as a representation of  $\mathrm{SL}_2(\mathbb{F}_r)$  is the same as  $\mathrm{Ind}_{\tilde{H}_2}^{\mathrm{SL}_2(\mathbb{F}_r)} \tilde{V}$ , where  $\tilde{V}$  is  $V$  regarded as a representation of  $\tilde{H}_2$ . Therefore, without loss of generality, we may (and do) assume  $G = \mathrm{SL}_2(\mathbb{F}_r)$ .

If  $H_1 \subset H_2 \subset G$ ,

$$\mathrm{Ind}_{H_2}^G \mathrm{Ind}_{H_1}^{H_2} V_1 = \mathrm{Ind}_{H_1}^G V_1,$$

so without loss of generality we may assume  $H$  is a maximal proper subgroup. Aside from the trivial representation,  $\mathrm{SL}_2(\mathbb{F}_r)$  has two irreducible representations of dimension  $\leq \frac{r-1}{2}$ , with characters  $\chi_{\beta}^{\pm}$ . As  $V$  is irreducible, it is a subrepresentation of the regular representation of  $H$ , and it follows that  $\mathrm{Ind}_H^G V$  is a subrepresentation of the regular representation of  $\mathrm{SL}_2(\mathbb{F}_r)$ . In particular,  $[G : H] \dim V$  reduces to 0 or 1  $(\bmod \frac{r-1}{2})$ , the former if  $\dim V > 1$ , and

$$[G : H] \dim V \leq 1^2 + (\dim \chi_{\beta}^{+})^2 + (\dim \chi_{\beta}^{-})^2 = \frac{r^2 - 2r + 3}{2} < \frac{|G|}{2(r+1)}.$$

It follows that  $|H| > 2(r+1)$ . By the classification of maximal subgroups of  $\mathrm{SL}_2(\mathbb{F}_r)$ , this means that  $H$  is a Borel subgroup  $B$  of  $G$  or that  $|H| \in \{24, 60, 120\}$ . The irreducible representations of  $B$  all have degree 1 or  $r-1$ . If  $\dim V = 1$ , the induced representation has degree  $r+1$ , which does not satisfy the congruence condition. If  $\dim V = r-1$ , the induced representation has degree  $r^2-1$ , which does not satisfy the inequality condition. This leaves a short list of possible triples  $(r, \dim V, |H|)$ . For  $r > 11$ , all can be ruled out by the congruence condition or the inequality condition. The only triples not ruled out are  $(7, 1, 48)$  and  $(11, 1, 120)$ . In each case, the degree of the induced representation is congruent to 1  $(\bmod \frac{r-1}{2})$ , so  $V$  must be trivial. By [At] pp. 3, 7, the induced representation in each case has an irreducible factor of degree  $r-1$ .  $\square$   $\square$

LEMMA 7: *Consider a short exact sequence of compact Lie groups*

$$0 \rightarrow G_1 \rightarrow G_2 \rightarrow \Gamma \rightarrow 0,$$

where  $\Gamma$  is  $\mathrm{SL}_2(\mathbb{F}_r)$  or  $\mathrm{PSL}_2(\mathbb{F}_r)$ ,  $r \geq 5$ . Suppose  $V$  and  $W$  are representations of  $G_2$ . If  $V \otimes W$  has

- (1) all its irreducible factors of degree  $\leq \frac{r-1}{2}$ ,
- (2) at least one irreducible factor of degree  $\frac{r-1}{2}$  which is  $G_1$ -scalar.
- (3) exactly one irreducible factor of degree 1,

then either  $V$  or  $W$  is one-dimensional.

*Proof:* Suppose first that  $V$  and  $W$  are irreducible and  $V \otimes W$  satisfies hypothesis (1). By Clifford's theorem, we can write  $V|_{G_1}$  and  $W|_{G_1}$  as direct sums of mutually conjugate isotypic representations  $V_i^m$  and  $W_j^n$  respectively. Let  $H_2 \supset G_1$  denote the subgroup of  $G_2$  stabilizing both  $V_1^m$  and  $W_1^n$ . Then  $\mathrm{Ind}_{H_2}^{G_2} V_1^m \otimes W_1^n$  is a  $G_2$ -subrepresentation of  $V \otimes W$ . By Lemma 6, it follows that  $V|_{G_1} \cong V_1^m$  and  $W|_{G_1} \cong W_1^n$ . (More generally, if any tensor product of  $G_2$ -representations satisfies (1), all of the irreducible constituents of all of the tensor factors are  $G_1$ -isotypic.)

By Lemma 3, replacing  $G_1$  and  $G_2$  by central extensions if necessary, we can write  $V$  as  $V_\sigma \otimes V_\tau$  and  $W$  as  $W_\sigma \otimes W_\tau$  with  $V_\sigma$  and  $W_\sigma$   $G_1$ -irreducible and  $V_\tau$  and  $W_\tau$   $G_1$ -scalar. As explained above, any irreducible constituent  $U$  of  $V_\sigma \otimes W_\sigma$  is isotypic for  $G_1$ ; passing to central extensions of  $G_1$  and  $G_2$  if necessary, we write  $U$  as  $U_\sigma \otimes U_\tau$ , so  $U_\sigma \otimes (U_\tau \otimes V_\tau \otimes W_\tau)$  is a  $G_2$ -subrepresentation of  $V \otimes W$ . Every irreducible factor of  $U_\tau \otimes V_\tau \otimes W_\tau$  is  $G_1$ -scalar. By Lemma 5, unless at least two of  $U_\tau$ ,  $V_\tau$ , and  $W_\tau$  have dimension 1, their tensor product has an irreducible factor of degree  $> \frac{r-1}{2}$  which is  $G_1$ -scalar. If  $\dim V_\tau = \dim W_\tau = 1$ ,

then  $V|_{G_1}$  and  $W|_{G_1}$  are irreducible; if  $X$  is any one-dimensional representation of  $G_1$ , then  $X^* \otimes V|_{G_1}$  is irreducible, so  $\dim(X^* \otimes V \otimes W)^{G_1} \leq 1$ .

Thus, if  $V \otimes W$  additionally satisfies hypothesis (2), either  $\dim V_\tau > 1$  or  $\dim W_\tau > 1$ . Without loss of generality, we assume the latter is true. Unless  $\dim U_\tau = \dim V_\tau = 1$ ,  $V \otimes W$  contains a factor which is the tensor product of the  $G_1$ -irreducible  $U_\sigma$  with a  $G_1$ -scalar irreducible of dimension  $> \frac{r-1}{2}$ . This is impossible by Lemma 4. The situation is therefore that  $V_\tau \otimes W_\tau$  is  $G_1$ -scalar,  $G_2$ -irreducible and of dimension  $\frac{r-1}{2}$ , and every factor of  $V_\sigma \otimes W_\sigma$  is  $G_1$ -irreducible. Applying (2) and Lemma 4 again, we see further that every  $G_2$ -irreducible factor of  $V_\sigma \otimes W_\sigma$  has dimension 1, so  $W_\sigma^* \otimes W_\sigma$  decomposes entirely into 1-dimensional pieces over  $G_2$ . If  $V'$  is any other irreducible representation such that  $V' \otimes W$  satisfies (1), then  $V'_\sigma$  is a twist of  $W_\sigma^*$  by a character so  $V'_\sigma \otimes V'^*_\sigma$  also decomposes as a sum of  $(\dim V'_\sigma)^2$  characters over  $G_2$ .

Now we return to the original problem. If  $V \otimes W$  satisfies all three hypotheses, there exists irreducible factors  $V_0$  and  $W_0$  of  $V$  and  $W$  respectively satisfying hypotheses (1) and (2). Without loss of generality, we assume  $\dim V_{0\tau} = 1$ . Then every irreducible factor of  $V$  is a twist of  $W_{0\sigma}^*$ , so they all have the same dimension  $d$ . By hypothesis (3), there exist irreducible factors  $V_i$  and  $W_j$  of  $V$  and  $W$  respectively such that  $V_i \otimes W_j$  has a one-dimensional  $G_2$ -submodule. Thus  $W_j^*$  must be a twist of  $V_i$ , and  $(V \otimes W_j)|_{G_1}$  decomposes entirely into  $d^2$  1-dimensional pieces. By (3),  $d = 1$ .

□

### §3. Case g=1

**Theorem 1.** The projective representation of  $M_1 = \text{Map}(S^1 \times S^1) = \text{SL}_2(\mathbb{Z})$  given by the  $\text{SO}(3)$ -theory at  $A = ie^{\frac{2\pi i}{4r}}$  is the same as the projective representation obtained by composing the  $(\text{mod } r)$  reduction map  $\text{SL}_2(\mathbb{Z}) \rightarrow \text{SL}_2(\mathbb{F}_r)$  with the odd factor of the Weil representation of  $\text{SL}_2(\mathbb{F}_r)$ .

We want explicit matrices for one of the two  $\frac{r-1}{2}$ -dimensional irreducible representations of  $\text{SL}_2(\mathbb{F}_r)$ . To find them, we briefly recall the theory of Weil representations over finite fields [Ge]. Let  $H_r$  denote the Heisenberg group of order  $r^3$ . We regard  $H_r$  as a central extension

$$0 \rightarrow \mathbb{F}_r \rightarrow H_r \rightarrow \mathbb{F}_r^2 \rightarrow 0.$$

The extension class defines a symplectic form (in this case, an area form) on the quotient. Any automorphism of  $H_r$  stabilizes the center and acts on  $\mathbb{F}_r^2$ , respecting this symplectic

form. Regarding  $\mathrm{SL}_2(\mathbb{F}_r)$  as the group of symplectic linear transformations of  $\mathbb{F}_r^2$ , we claim that its action lifts to  $H_r$ . To make this explicit, let  $x$  and  $y$  be elements of  $H_r$  whose images in  $\mathbb{F}_r^2$  form a unimodular basis. Let  $z$  be the generator of the center defined by  $z^2 = yxy^{-1}x^{-1}$ . Finally, for  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_r)$ , let

$$f_M(x) = z^{ac}x^a y^c, \quad f_M(y) = z^{bd}x^b y^d, \quad f_M(z) = z.$$

We easily check that this defines an action of  $\mathrm{SL}_2(\mathbb{F}_r)$  on  $H_r$ .

Let  $e(k)$  denote  $e^{\frac{2\pi i k}{r}}$ . Let  $(\rho, V)$  denote the Stone-von Neumann representation (i.e., the unique irreducible representation of  $H_r$  with central character  $z^k \mapsto e(k)$ .) We fix a basis  $e_0, \dots, e_{r-1}$  for  $V$  so that  $e_k$  is the  $e(2k)$ -eigenspace of  $x$ . In this basis, we write

$$\rho(x) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & e(2) & 0 & \cdots & 0 \\ 0 & 0 & e(4) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e(-2) \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \rho(z) = e(1) \mathrm{Id}.$$

For any  $\alpha \in \mathrm{SL}_2(\mathbb{F}_r) \subset \mathrm{Aut}(H_r)$ ,  $\rho \circ \alpha$  is equivalent to  $\rho$ . There exists  $R_\alpha$ , therefore, unique up to scalar multiples, such that

$$R_\alpha \rho(h) R_\alpha^{-1} = \rho(\alpha(h)) \quad (*)$$

for all  $h \in H_r$ . If  $\bar{R}_\alpha$  denotes the class of  $R_\alpha$  in  $\mathrm{PGL}(V)$ , we conclude that  $\alpha \mapsto \bar{R}_\alpha$  is a projective representation. When  $r \geq 5$ ,  $\mathrm{SL}_2(\mathbb{F}_r)$  is perfect and centrally closed, so there is a unique lifting to an  $r$ -dimensional linear representation, which we call the **Weil representation** of  $\mathrm{SL}_2(\mathbb{F}_r)$ . Explicitly we may choose (up to scalar multiplication)

$$R_S = (e(2ij))_{0 \leq i,j < r}, \quad R_T = (\delta_{ij} e(-i^2))_{0 \leq i,j < r}.$$

We can verify  $(*)$  by checking it for  $h = x$  and  $h = y$ .

Let  $E = \{0, 2, 4, \dots, r-3\}$  and set

$$f_i = e_{\frac{r-1-i}{2}} - e_{\frac{r+1+i}{2}}, \quad i \in E.$$

It is easy to see that the span of  $f_0, f_2, f_4, \dots, f_{r-3}$  forms an invariant subspace  $V^{\mathrm{odd}}$  of both  $R_S$  and  $R_T$ . In terms of this basis,  $R_S$  is represented by the matrix

$$R_S^{\mathrm{odd}} = \left( e \left( 2 \left( \frac{r-1-i}{2} \right) \left( \frac{r-1-j}{2} \right) \right) - e \left( -2 \left( \frac{r-1-i}{2} \right) \left( \frac{r-1-j}{2} \right) \right) \right)_{i,j \in E},$$

and  $R_T$  is represented by

$$R_T^{\text{odd}} = \left( \delta_{ij} e \left( - \left( \frac{r-1-j}{2} \right)^2 \right) \right)_{i,j \in E}.$$

As

$$A^2 = e \left( \frac{r+1}{2} \right),$$

we obtain

$$R_S^{\text{odd}} = \left( A^{2(i+1)(j+2)} - A^{-2(i+1)(j+1)} \right)_{i,j \in E} = (A^2 - A^{-2}) \left( [(i+1)(j+1)]_A \right)_{i,j \in E},$$

and

$$R_T^{\text{odd}} = e^{-\frac{2\pi i(r-1)^2}{4r}} \left( \delta_{ij} A^{-j(j+2)} \right)_{i,j \in E}.$$

Thus the composition  $\text{SL}_2(\mathbb{Z}) \rightarrow \text{SL}_2(\mathbb{F}_r) \rightarrow \text{PGL}(V^{\text{odd}})$  is equivalent to  $\rho_{\text{SO}(3)}$ .  $\square$

#### §4. Case g=2

In what follows, we write  $X_r$  for

$$X_r = \left\{ e^{\frac{2\pi i n^2}{r}} \mid 0 < n < \frac{r}{2} \right\}.$$

LEMMA 8: *Let  $G$  be a simple compact Lie group with Coxeter number  $h$ ,  $r \geq 5$  a prime, and  $\rho : G \rightarrow \text{GL}(V)$  an irreducible representation of  $G$  such that  $r$  divides  $\dim V$ . If there exists  $g \in G$  such that the spectrum of  $\rho(g)$  is  $X_r$ , then  $r \leq 2h - 5$ .*

*Proof:* Let  $T$  be a maximal torus containing  $g$  and  $(\cdot, \cdot)$  denote the Cartan-Killing form on the character space  $X^*(T) \otimes \mathbf{R}$ . Let

$$\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)},$$

and fix a Weyl chamber. If  $V$  has highest weight  $\lambda$  and  $\rho$  is the half sum of positive roots, the Weyl dimension formula ([Bo] VIII, §9, Th. 2) asserts

$$\dim V = \prod_{\alpha > 0} \frac{\langle \lambda, \alpha \rangle + \langle \rho, \alpha \rangle}{\langle \rho, \alpha \rangle},$$

where the product is taken over all positive roots. Let  $\beta$  denote the highest root. Since  $r$  divides  $\dim V$ , and  $\langle \mu, \alpha \rangle \in \mathbb{Z}$  for all weights  $\mu$  and roots  $\alpha$ ,

$$\langle \lambda, \beta \rangle + \langle \rho, \beta \rangle \geq r,$$

or, by [Bo] VI, §1, Prop. 29(c) and [Bo] VI, §1, Prop. 31,

$$\langle \lambda, \beta \rangle + h - 1 \geq r.$$

The string of weights  $\lambda, \lambda - \beta, \dots$  has length  $1 + \langle \lambda, \beta \rangle$  ([Bo] VIII, §7, Prop. 3(i)). For any  $w$  in the Weyl group, the string  $w(\lambda), w(\lambda) - w(\beta), \dots$  has the same length. The Weyl orbit of  $\beta$  consists of all long roots ([Bo] VI, §1, Prop. 11), so the lattice it generates contains the root lattice if  $G$  is of type A, D, or E; twice the root lattice if  $G$  is of type B, C, or F; and three times the root lattice if  $G$  is of type G ([Bo] Planches). Since the difference between weights in an irreducible representation belongs to the root lattice, and since the eigenvalues of  $\rho(g)$  are  $r$ th roots of unity, not all equal, we conclude that if  $r \geq 5$ ,  $w(\beta)(g)$  is a primitive  $r$ th root of unity for some  $w \in W$ . A non-trivial geometric progression in  $X_r$  has length  $\leq \frac{r-1}{2}$ , so

$$1 + \langle \lambda, \beta \rangle \leq \frac{r-1}{2}.$$

Thus,  $r \leq 2h - 5$ .  $\square$

LEMMA 9: *Under the hypotheses of Lemma 8, if  $\dim V = \frac{r^3-r}{24}$ , then  $G$  is a classical group. If  $V$  is not self-dual, then  $G$  is of type  $A_n$ .*

*Proof:* The Coxeter numbers of  $G_2$ ,  $F_4$ ,  $E_6$ ,  $E_7$ ,  $E_8$  are 6, 12, 12, 18, 30, respectively ([Bo] Planches). Examination of all primes  $\leq 55$  [MP] reveals that the only case in which  $\frac{r^3-r}{24}$  is the dimension of a representation of a suitable exceptional group is  $r = 7$ . The group is  $G_2$ , and  $V$  is the adjoint representation. This case is excluded, however, as the longest string of short roots is  $4 > \frac{r-1}{2}$ . Thus  $\beta(g) = 1$  for all short roots  $\beta$ . As the short roots generate the root lattice,  $g$  has all eigenvalues equal, contrary to assumption.

If  $V$  is not self-dual,  $G$  cannot be of type  $B_n$  or  $C_n$  and can only be of type  $D_n$  if  $n$  is odd and the highest weight  $\lambda = a_1\lambda_1 + \dots + a_n\lambda_n$  satisfies  $\sup(a_{n-1}, a_n) > 0$  [MP]. As the Weyl dimension formula is monotonic in each  $a_i$ ,  $\dim V \geq 2^{n-1}$  while  $r \leq 2h - 5 = 4n - 9$ . Given the dimension of  $V$ , the only possibilities for  $(n, r)$  are  $(5, 11)$ ,  $(7, 13)$ ,  $(7, 17)$ ,  $(7, 19)$ ,  $(9, 19)$ ,  $(9, 23)$ ,  $(11, 31)$ . For  $r \leq 31$ , the longest geometric progression in  $X_r$  has length  $\leq 4$ , so  $h \geq r - 2$  or  $r \leq 2n$ . This leaves only the case  $(7, 13)$ , and by [MP], the only irreducible 91-dimensional representation of  $D_7$  is self-dual.  $\square$

LEMMA 10: *For  $n \geq 11$ , if  $G = \mathrm{SU}(n)$  and  $V$  is an irreducible representation of  $G$  with highest weight  $\lambda$  and dimension  $< 2\binom{n}{3}$ , then  $\lambda$  is one of the following:*

<i>Highest weight</i>	<i>Dimension</i>
0	1
$\lambda_1, \lambda_{n-1}$	$n$
$\lambda_2, \lambda_{n-2}$	$\binom{n}{2}$
$2\lambda_1, 2\lambda_{n-1}$	$\binom{n+1}{2}$
$\lambda_1 + \lambda_{n-1}$	$n^2 - 1$
$\lambda_3, \lambda_{n-3}$	$\binom{n}{3}$
$3\lambda_1, 3\lambda_{n-1}$	$\binom{n+2}{3}$

*Proof:* By the monotonicity of the Weyl dimension formula, it is enough to check that the dimension is always greater than  $2\binom{n}{3}$  in the following cases (and their duals):  $\lambda_k$ ,  $4 \leq k \leq n-4$ ;  $4\lambda_1$ ;  $2\lambda_2$ ;  $2\lambda_3$ ;  $\lambda_1 + \lambda_2$ ;  $\lambda_1 + \lambda_3$ ;  $\lambda_2 + \lambda_3$ ;  $\lambda_1 + \lambda_{n-2}$ ;  $\lambda_1 + \lambda_{n-3}$ ;  $\lambda_2 + \lambda_{n-2}$ ;  $\lambda_2 + \lambda_{n-3}$ ;  $\lambda_3 + \lambda_{n-3}$ ;  $2\lambda_1 + \lambda_{n-1}$ .  $\square$

LEMMA 11: *Let  $r \geq 7$  be a prime,  $G$  a simple compact Lie group, and  $\rho : G \rightarrow \mathrm{GL}(V)$  an irreducible representation of  $G$  such that*

- 1)  *$V$  is not self dual,*
- 2)  *$\dim V = \frac{r^3 - r}{24}$ ,*
- 3) *There exists  $g \in G$  such that the spectrum of  $\rho(g)$  is  $X_r$ .*

*Then either  $G = \mathrm{SU}(\dim V)$ , and  $V$  is the standard representation or its dual; or  $r = 13$ ,  $G$  is a central quotient of  $\mathrm{SU}(14)$ , and  $V$  is the exterior square representation or its dual.*

*Proof:* By Lemma 8,  $r \leq 2h - 5$ . By Lemma 9,  $G$  is of type  $A_n$ , so  $n \geq \frac{r+3}{2}$ , so

$$2\binom{n}{3} \geq \frac{(r+3)(r+1)(r-1)}{24} > \frac{r^3 - r}{24}.$$

If  $n+1 < 11$ ,  $r < 17$ , so there are three cases:  $r = 7$ ,  $n \geq 5$ , and  $\dim V = 14$ ;  $r = 11$ ,  $n \geq 7$ , and  $\dim V = 55$ ; and  $r = 13$ ,  $n \geq 8$ , and  $\dim V = 91$ . For  $n \leq 10$ , we see there is just one possibility for an irreducible representation of  $\mathrm{SU}(n+1)$  of the given dimension:  $r = 11$ ,  $n = 10$ , and  $V$  is the exterior square of the standard representation of  $\mathrm{SU}(11)$  or its dual. If  $n+1 \geq 11$ , by Lemma 10, either  $V$  or  $V^*$  has highest weight in the set

$$\{\lambda_1, 2\lambda_1, 3\lambda_1, \lambda_2, \lambda_3, \lambda_1 + \lambda_{n-1}\}$$

As  $V$  is not self-dual, we can exclude  $\lambda_1 + \lambda_{n-1}$ . For  $r \geq 7$ ,

$$\frac{(r+1)r(r-1)}{24} < \frac{(r-1)(r-2)(r-3)}{8},$$

so  $\binom{m}{3} \leq \frac{r^3 - r}{24}$  only when  $m < r - 1$ , in which case equality is ruled out since  $r$  does not divide  $\binom{m}{3}$ . Applying this when  $m = n \pm 1$ , we exclude the cases  $\lambda_3$  and  $3\lambda_1$ . For  $\lambda_2$  and  $2\lambda_1$ , we seek solutions of

$$\binom{m}{2} = \frac{r^3 - r}{24}$$

in integers  $m$ . For any such solution,  $r|m$  or  $r|m - 1$ , so

$$12a(ar \pm 1) = r^2 - 1, \quad a \in \mathbb{N}.$$

The discriminant of the quadratic equation for  $r$  is  $(12a^2)^2 + 4 \pm 48a$ , and

$$(12a^2 - 1)^2 < (12a^2)^2 + 4 - 48a < (12a^2)^2 < (12a^2)^2 + 4 + 48a < (12a^2 + 1)^2$$

for  $a \geq 3$ . For  $a = 2$ , the discriminant is not square for either choice of sign. For  $a = 1$ , we get the two solutions  $(r, m) = (11, 11)$  and  $(r, m) = (13, 14)$ . An exhaustive analysis of sets  $S$  of  $r$ th roots of unity, whose symmetric or exterior squares give  $X_{11}$  or  $X_{13}$  reveals exactly two possibilities: the set

$$\{1, \zeta_{13}, \zeta_{13}^3, \zeta_{13}^9\} \tag{1}$$

and its complex conjugate have exterior square  $X_{13}$ .  $\square$

Let  $\rho_g : M_g \rightarrow \mathrm{PGL}(V_g)$  denote the projective unitary representation given by the  $\mathrm{SO}(3)$ -theory. Let  $G_g$  denote the closure of the image. It is a subgroup of  $\mathrm{PSU}(\dim V_g)$  and therefore a compact Lie group. We will often regard  $V_g$  as a projective representation of  $G_g$ .

**Theorem 2.** For  $g = 2$ , the projective representation  $\rho$  associated to the  $\mathrm{SO}(3)$  theory at  $A = ie^{\frac{2\pi i}{4r}}$  for  $r \geq 5$  has dense image.

The proof will be carried out in several steps. By [FLW2], we may assume  $r \geq 7$ .

**Step 1.** The Lie group  $G_2$  is infinite.

*Proof:* Consider the decomposition of the representation space arising from a curve separating a genus 2 surface into two genus 1 surfaces with boundary. The components are indexed by labels  $0, 2, \dots, r - 3$ , and they are projective representation spaces of  $M_1 \times M_1$ . The representation associated with label  $2l$  is of the form  $W_{2l} \otimes W_{2l}$ , where each tensor factor has dimension  $\frac{r-1-2l}{2}$ . For label  $r - 5$ , it has dimension 2. Thus we have a two-dimensional projective unitary representation of  $M_1 = \mathrm{SL}_2(\mathbb{Z})$ ; the ratio of eigenvalues for a Dehn twist is a primitive  $r$ th root of unity. By the classification of finite subgroups of  $\mathrm{SO}(3)$ , this implies the image is infinite, and it follows that the same is true for  $G_2$ .

**Step 2.** The projective representation  $V_2$  is not self-dual.

*Proof:* Equivalently, for any central extension  $\tilde{M}_2$  for which one can lift  $V_2$  to a linear representation (also denoted  $V_2$ ), the contragredient representation  $V_2^*$  is not obtained by tensoring  $V_2$  by a central character. We compute the multiplicities of the eigenvalues of a lift to  $\tilde{M}_2$  of a Dehn twist. These are just the dimensions  $d_{r,1,2l,2l}$  of a doubly-punctured torus with both labels equal to  $2l$ , and are therefore given by

$$d_{r,1,2l,2l} = \frac{(2l+1)(r-2l-1)}{2}. \quad (3)$$

No two of these multiplicities coincide as  $l$  ranges over integers  $\leq \frac{r-3}{2}$ , so  $V_2$  cannot be self-dual.

**Step 3.** Let  $\tilde{G}_2$  denote any central extension of  $G_2$  for which  $V_2$  lifts to a linear representation (which we also denote  $(\rho_2, V_2)$ ). Let  $\tilde{G}_2^\circ$  denote the identity component of  $\tilde{G}_2$ . Then the restriction of  $V_2$  to  $\tilde{G}_2^\circ$  is isotypic.

*Proof:* As  $M_2$  is generated by Dehn twists, some lift  $\tilde{t} \in \tilde{G}_2$  of a Dehn twist  $t$  would otherwise permute the isotypic components non-trivially. Thus, the eigenvalues of  $\rho_2(\tilde{t})$  (which are defined up to multiplication by a common scalar) would contain a coset of a non-trivial group of roots of unity [FLW2] Lemma 1.2. This is impossible, since up to scalars, the spectrum of a Dehn twist is  $X_r$ .

**Step 4.** For any central extension  $\tilde{G}_2$  of  $G_2$  as above and any normal subgroup  $\tilde{G}'_2$  such that every homomorphism  $\mathrm{SL}_2(\mathbb{F}_r) \rightarrow \tilde{G}_2/\tilde{G}'_2$  is trivial,  $V_2$  is tensor indecomposable as a  $\tilde{G}'_2$ -representation.

*Proof:* The restriction of  $V_2$  to

$$\tilde{M}_1 = \{1\} \widetilde{\times} M_1 \subset M_1 \widetilde{\times} M_1 \subset \tilde{M}_2$$

decomposes as a sum of terms of the form  $W_{2l}$ . Let  $\tilde{H}_1$  denote the closure of  $\tilde{M}_1$  in  $\mathrm{GL}(V_2)$  and  $\tilde{H}'_1$  the intersection of  $\tilde{H}_1$  with  $\tilde{G}'_2$ . The occurrence of  $W_0$  as a factor guarantees that  $\mathrm{SL}_2(\mathbb{F}_r)$  or  $\mathrm{PSL}_2(\mathbb{F}_r)$  is a quotient of  $\tilde{H}_1$  for which the  $W_0$ -factor in question is an irreducible representation of degree  $\frac{r-1}{2}$ ; the condition on  $\tilde{G}'_2$  guarantees that  $\tilde{H}'_1$  maps onto  $\mathrm{SL}_2(\mathbb{F}_r)$ , so  $V_2|_{\tilde{H}'_1}$  has an irreducible factor which is the composition of this quotient map and a degree  $\frac{r-1}{2}$  representation of  $\mathrm{SL}_2(\mathbb{F}_r)$ . Tensor indecomposability now follows from Lemma 7.

**Step 5.** The restriction of  $V_2$  to  $G_2^\circ$  is irreducible.

*Proof:* Otherwise, by Lemma 3, there exists a central extension  $\tilde{G}_2$  of  $G_2$  so that regarding  $V_2$  as a  $\tilde{G}_2$ -representation, it has a tensor-decomposition.

**Step 6.** The identity component  $G_2^\circ$  is a simple compact Lie group.

*Proof:*

The identity component  $K = G_2^\circ$  is a connected compact Lie group. As  $V_2$  is an irreducible projective representation, the center of  $K$  is trivial. Therefore, it is a product of compact simple Lie groups  $K_1 \times \cdots \times K_s$ , and  $V_2$  is a tensor product of unitary projective representations  $X_1, \dots, X_s$  of the  $K_i$ . In other words,

$$K_1 \times \cdots \times K_s \hookrightarrow \mathrm{PSU}(X_1) \times \cdots \times \mathrm{PSU}(X_s) \hookrightarrow \mathrm{PSU}(V_2),$$

where the first inclusion is the product of inclusion  $K_i \hookrightarrow \mathrm{PSU}(X_i)$  and the second is the tensor product map. As  $2^r > \frac{r^3-r}{24}$  for all  $r, s < r$ . Consider the composition  $\pi : G_2 \rightarrow \mathrm{Aut}(K) \rightarrow \mathrm{Out}(K)$ . The outer automorphism group is contained in a product of groups of the form  $\mathrm{Out}(K_i)^{s_i} \rtimes S_{s_i}$ , where  $s_i \leq s$ . The largest proper subgroup of  $\mathrm{SL}_2(\mathbb{F}_r)$  is the Borel subgroup with index  $r+1 > s_i$ , so any homomorphism  $\mathrm{SL}_2(\mathbb{F}_r) \rightarrow S_{s_i}$  is trivial. Therefore, any homomorphism from  $\mathrm{SL}_2(\mathbb{F}_r)$  to  $\mathrm{Out}(K)$  lands in a product of solvable groups, and since  $\mathrm{SL}_2(\mathbb{F}_r)$  is perfect, that means any such homomorphism is trivial. By Step 4,  $V_2$  is tensor indecomposable as a representation of any central extension of the subgroup  $G'_2 = \ker \pi$ . However,  $g$  acts by inner automorphisms on  $K$  for  $g \in G'_2$ , and since  $\rho_2$  is irreducible, this means  $G'_2 \subset K \subset \prod_{i=1}^s \mathrm{PSU}(X_i)$ , so  $s = 1$ .

**Step 7.** The theorem holds if  $r \neq 13$ .

*Proof:* Applying Lemma 11 to the universal cover of  $G_2^\circ$ ,  $G_2^\circ$  is all of  $\mathrm{PSU}(\dim V_2)$ , so the same is true for  $G_2$ .

**Step 8.** The theorem holds if  $r = 13$ .

*Proof:*

For  $r = 13$ , we must consider the possibility that the universal covering group of  $G_2^\circ$  is  $\mathrm{SU}(14)$ ,  $V_2$  is its alternating square, and a Dehn twist has exactly four different eigenvalues  $\lambda_i$  in  $\mathrm{SU}(14)$  given up to a common scalar multiple by (1) or its complex conjugate. By (3), the eigenvalues of a Dehn twist have multiplicities 6, 15, 20, 21, 18, 11 in  $\mathrm{PSU}(\dim V_2)$ , and each one arises uniquely as a product of distinct eigenvalues  $\lambda_i$ . Therefore, some  $\lambda_i$  must have multiplicity 11, but this is impossible since only one of the eigenvalues in  $V_2$  has multiplicity divisible by 11. Therefore,  $G_2^\circ = \mathrm{PSU}(\dim V_2)$  also for  $r = 13$ .

□

## §5. Case $g \geq 3$

**Theorem 3.** For all  $r \geq 5$  and all  $g \geq 2$ ,  $\rho_g(M_g)$  is dense in  $\mathrm{PSU}(\dim V_g)$ .

For  $r = 5$ , this is already known [FLW2] Theorem 6.2. We therefore assume from now on that  $r \geq 7$ . We begin with a dimension estimate.

LEMMA 12: *For  $r \geq 7$  and  $g \geq 2$ ,*

$$\dim V_{\Sigma_{g+1}} < \binom{\dim V_{\Sigma_g}}{2} \quad (2)$$

*except when  $r = 7$  and  $g = 2$ .*

*Proof:* As  $\alpha_k > 1$  for all  $k$ , if  $g \geq 3$ ,

$$\dim V_{\Sigma_{g+1}} = \sum_k \alpha_k^g \leq \sum_k \alpha_k^{3(g-1)/2} < \dim^{3/2} V_{\Sigma_g} < \binom{\dim V_{\Sigma_g}}{2}.$$

For  $g = 2$ , we compute

$$\binom{\dim V_{\Sigma_2}}{2} - \dim V_{\Sigma_3} = \frac{(r+5)(r+3)(r+1)r(r-1)(r-8)}{5760},$$

and this quantity is obviously positive when  $r > 7$ .  $\square$

*Proof of Theorem 3.*

The proof is very similar to that of Theorem 2.

**Step 1.** The Lie group  $G_g$  is infinite.

*Proof:* Consider the decomposition of the representation space arising from a curve separating a genus  $g$  surface into two pieces, one of genus 1 and one of genus  $g-1$ . Restricting to  $M_{g-1} \times M_1$ , we obtain a decomposition

$$V_g = \bigoplus_{l=0}^{\frac{r-3}{2}} X_{g-1,2l} \otimes W_{2l},$$

where  $X_{g-1,2l}$  denotes the projective representation space of  $M_{g-1}$  associated to a surface of genus  $g-1$  with a single boundary component labeled  $2l$ . Now we proceed as in Step 1 of Theorem 2.

**Step 2.** The projective representation  $V_g$  is not self-dual.

*Proof:* If  $V_g$  is self-dual, its restriction to  $M_{g-1} \times M_1$  decomposes into self-dual projective representations and mutually dual pairs of projective representations. We use induction on  $g$ , the base case being Step 2 of Theorem 2. By the induction hypothesis,  $X_{g-1,0} \otimes W_0 = V_{g-1} \otimes W_0$  is not self-dual. Neither can it be dual to any other factor since  $W_0 = V_1$  is irreducible and the other representations  $W_{2l}$  have lower dimension.

**Step 3.** Let  $\tilde{G}_g$  denote any central extension of  $G_g$  for which  $V_g$  lifts to a linear representation. Let  $\tilde{G}_g^\circ$  denote the identity component of  $\tilde{G}_g$ . Then the restriction of  $V_g$  to  $\tilde{G}_g^\circ$  is isotypic.

*Proof:* Identical to the proof of Step 3 of Theorem 2.

**Step 4.** For any central extension  $\tilde{G}_g$  of  $G_g$  as above and any normal subgroup,  $V_g$  is tensor indecomposable as a  $\tilde{G}_g$ -representation.

*Proof:* The restriction of  $V_g$  to

$$\tilde{M}_1 = \{1\} \widetilde{\times} M_1 \subset M_{g-1} \widetilde{\times} M_1 \subset \tilde{M}_g$$

decomposes as a sum of terms of the form  $W_{2l}$ . Now we proceed as in Step 4 of Theorem 2.

**Step 5.** The restriction of  $V_g$  to  $G_g^\circ$  is irreducible.

*Proof:* Identical to the proof of Step 5 of Theorem 2.

**Step 6.** The identity component  $G_g^\circ$  is a simple compact Lie group.

*Proof:* Let  $K = G_g^\circ$ . As in Step 6 of Theorem 2,  $K$  is a product of compact simple Lie groups  $K_i$ , and  $V_g$  is the tensor product of unitary projective representations  $X_i$  of the  $K_i$ . Thus,

$$K_1 \times \cdots \times K_s \hookrightarrow \mathrm{PSU}(X_1) \times \cdots \times \mathrm{PSU}(X_s) \hookrightarrow \mathrm{PSU}(V_g).$$

The conjugation action of  $G_2$  on  $K$  must act transitively on the factors, since any decomposition into orbits gives a tensor decomposition of  $V$ . Therefore, the  $K_i$  are mutually isomorphic, and their representations  $X_i$  are equivalent up to composition with automorphisms of  $X_i$ ; in particular, their degrees are all the same. Now, the closure of  $M_{g-1} \subset M_{g-1} \times M_1$  in  $K$  maps onto  $G_{g-1}$  since  $V_{g-1} = X_{g-1,0}$  is a summand of the restriction of  $V_g$  to  $M_{g-1}$ . Thus some factor  $K_i$  maps onto  $G_{g-1}$ . By the induction hypothesis,

$$\dim K_i \geq \dim G_{g-1} = \dim V_{g-1}^2 - 1,$$

so any non-trivial representation of  $K_i$  has degree  $\geq \dim V_{g-1}$ . Therefore,  $\dim V_g \geq \dim V_{g-1}^s$ . The inequality (2) then implies  $s = 1$  except possibly when  $g = 3$  and  $r = 7$ , in which case,  $\dim V_g = 98 < 14^2 = \dim V_{g-1}^2$ , so again  $s = 1$ .

**Step 7.** For all  $g \geq 3$  and  $r \geq 7$ ,  $G_g = \mathrm{PSU}(\dim V_g)$ .

*Proof:* We use induction, the base case being Theorem 2. By the induction hypothesis and (2),

$$\mathrm{rank} G_g^\circ \geq \mathrm{rank} G_{g-1} \geq \dim V_{g-1} - 1 > \sqrt{2 \dim V_g + 1/4} - 1/2. \quad (4)$$

By Step 2,  $V_g$  is not self-dual, so by [MP],  $G_g$  is of type  $A_n$ ,  $D_n$  ( $n$  odd), or  $E_6$ . The case  $E_6$  is ruled out since  $\dim V_{g-1} > 7$  in all cases  $g \geq 3$ . The minimal dimension for a representation of  $D_n$  which is not self-dual is  $2^{n-1} > \binom{n+1}{2}$  for  $n > 4$ . This leaves the case  $A_n$ , where the inequality  $\dim V_g < \binom{n+1}{2}$  implies  $\dim V_g < 2\binom{n+1}{3}$  for  $n > 2$ . Lemma 10 gives the list of possibilities. For  $n > 7$ , the possible highest weights are  $\lambda_1, \lambda_n, \lambda_2, \lambda_{n-1}, 2\lambda_1, 2\lambda_n$ , and  $\lambda_1 + \lambda_n$ , the last being ruled out as  $V_g$  is not self-dual. Up to duality, then,  $V_g$  is either the standard representation, its exterior square, or its symmetric square, and both are ruled out by (4).

## §6. An application

Besides determining the representations of the mapping class groups, a TQFT also determines invariants of oriented closed 3-manifolds [RT]. To describe the  $\mathrm{SO}(3)$  invariant of 3-manifolds, we introduce the following notations: let  $d_i$  and  $\theta_i$  be the quantum dimension and the twist of the label  $i$ , and  $D$  the global dimension of the modular tensor category defined in Section 1; then we define  $p_\pm = \sum_{i \in L} \theta_i^{\pm 1} d_i^2$ , and  $\omega_0$  to be the formal sum  $\sum_{i \in L} \frac{d_i \cdot i}{D}$ . If a 3-manifold  $M^3$  is represented by a framed link  $L$  in  $S^3$ , then the  $\mathrm{SO}(3)$  3-manifold invariant of  $M^3$  is  $\tau(M^3) = \frac{1}{D} \cdot \langle \omega_0 * L \rangle \cdot (\frac{p_-}{D})^{\sigma(L)}$ , where  $\sigma(L)$  is the signature of the linking matrix of  $L$ , and  $\langle \omega_0 * L \rangle$  is the link invariant of  $L$  where each component of  $L$  is labeled by  $\omega_0$  [T]. Note that in this normalization,  $\tau(S^1 \times S^2) = 1, \tau(S^3) = \frac{1}{D}$ . The invariant  $\tau$  is multiplicative for disjoint unions, but for connected sums  $\tau(M_1 \# M_2) = D \cdot \tau(M_1) \tau(M_2)$ .

Recall that a TQFT 3-manifold invariant is only defined for extended oriented closed 3-manifolds, but as pointed out in [A] there is a preferred framing for each oriented closed 3-manifold; therefore, the formula above should be thought as for an extended 3-manifold with the preferred framing determined by the framed link  $L$ . The same 3-manifold invariant can also be defined using the representations of the mapping class groups. The subtlety in framing is reflected in the fact that the TQFT representations of the mapping class groups are only projective representations. It is known that  $\frac{p_-}{D}$  is a root of unity of finite order

[BK], so we can write  $\frac{p_-}{D} = e^{\frac{\pi i c}{4}}$  for a rational number  $c$ , which is called the central charge of the TQFT (well-defined modulo 8). Framing changes lead to powers of  $\kappa = e^{\frac{\pi i c}{4}}$ . So up to powers of  $\kappa$ , the same  $SO(3)$  invariant of 3-manifolds can be obtained as follows: suppose an oriented 3-manifold  $M^3$  is given by gluing together two genus- $g$  oriented handlebodies  $H_g$  by a self-diffeomorphism  $f$  of  $\Sigma_g = \partial H_g$  (note that the handlebody  $H_g$  determines a vector  $v_0$  in the TQFT vector space  $V_r(\Sigma_g)$  of  $\Sigma_g$  up to a power of  $\kappa$ ); then the  $SO(3)$  3-manifold invariant  $\tau(M^3)$  is, up to a power of  $\kappa$ , the inner product of  $v_0$  with  $\rho_{SO(3)}(f)(v_0)$ . Theorem 3 has a direct corollary concerning the Witten-Reshitikhin-Turaev  $SO(3)$  invariants of 3-manifolds.

**Theorem 4.** The set of the norms of the  $SO(3)$  invariants at  $A = ie^{\frac{2\pi i}{4r}}$  of all connected 3-manifolds is dense in  $[0, \infty)$  for primes  $r \geq 5$ .

*Proof:* Given any complex number  $z$ , note that  $D > 1$  so we can find a  $g > 1$  so that  $z' = \frac{z}{D^{g-1}}$  satisfies  $|z'| < 1$ . Then arrange  $z'$  as the  $(1, 1)$  entry of a unitary matrix  $U_z$ , and the vector  $v_0$  determined by  $H_g$  as the first basis vector of an orthonormal basis of  $V_r(\Sigma)$ . By Theorem 3,  $U_z$  can be approximated by a sequences of unitary matrices associated to diffeomorphisms  $f_i$  of  $\Sigma_g$  up to a phase. Each  $f_i$  determines a 3-manifold  $M_{f_i}$  by gluing two copies of  $H_g$ . It follows that the 3-manifold invariant of  $M_{f_i}$  is the  $(1, 1)$ -entry of  $\rho_{SO(3)}(f_i)$  up to a phase, hence approximates  $z'$  up to a phase. By connecting sums  $M_{f_i}$  with  $(g-1)$  copies of  $S^1 \times S^2$ , we approximate  $z$  using the invariants of the 3-manifolds  $M_{f_i} \# (g-1)(S^1 \times S^2)$ .  $\square$

*Remarks:* (1): A similar density result of the  $SU(2)$  invariants for  $r = 1 \bmod 4$  can be deduced from the  $SO(3)$  case and the decomposition formula.

(2): This result does not follow from Funar's result [F]. The infinite mapping classes are in the image of the braid groups, and can be extended to the handlebody groups of  $H_g$ . Therefore, all resulting 3-manifolds have the same invariant up to powers of  $\kappa$ .

Finally, we make the following:

**Conjecture:** The set of the  $SO(3)$  invariants at  $A = ie^{\frac{2\pi i}{4r}}$  of all connected 3-manifolds is dense in the complex plane for primes  $r \geq 5$ .

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